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Global stabilization of a four rotor helicopter with bounded inputs

Ahmad Hably and Nicolas Marchand

Abstract—This paper proposes a global asymptotic stabilizing control law for a quad-rotor helicopter with bounded inputs. The proposed control design exploits the technique based on the sum of saturating functions and is based on the global stabilization of multiple integrators with bounded inputs. The positiveness of the thrust and the boundedness of the control inputs are taken into account. Numerical simulations show the effectiveness of the proposed controller.

I. INTRODUCTION

The control and design of mini aerial robots have received much attention within the automatic control community throughout the last decade. This interest was motivated by the enormous military and civil applications of such vehicles accompanied with the technological progress in sensors, actuators, processors and power storage devices. Within mini aerial robots, the quad-rotor helicopter (also known as the X-4 flyer) that is a helicopter with four fixed rotors and is one of the most interesting architecture because of its mechanical simplicity. Moreover, this under-actuated dynamical system is characterized by the payload augmentation and a high maneuverability [1]. The complete model of this quad-rotor has been the subject of several papers (see for instance [2], [14] or [15]). Fig.1 depicts the quad-rotor developed at gipsa-lab in Grenoble within the "Drone" project.



Fig. 1. The quad-rotor helicopter prototype of the gipsa-lab

Different control schemes have been explored. For the quad-rotor's rotational dynamics, PID and LQ control techniques are studied in [3]. Sliding mode control is utilized in [4] and [20]. Backstepping control is also applied to stabilize the orientation (limiting it to small values) and the height of the quad-rotor helicopter in [7]. These approaches, focused on the rotational dynamics, do not consider neither the boundedness and the positiveness of the thrust nor the translational dynamics of the quad-rotor helicopter. In fact,

actuators saturation has a significant effect on the overall stability of the aircraft [6]. To overcome this constraint, several tools have been introduced for analyzing and controlling linear and sometimes also nonlinear systems with bounded inputs. However, strategies to handle actuators constraints for autonomous UAV's (Unmanned Aerial Vehicles) are very few. In [10], a gain scheduling approach is applied to stabilize a pitch axis flight in presence of input rate saturation. The proposed approach is based on recent results on stabilization of linear systems by means of nonlinear bounded feedbacks [12], [17], [18], [19]. Nested saturation algorithms first proposed by [19] has already been used in [9] extending results previously given in [5] for the quad-rotor helicopter. In these works, the model is obtained via a Lagrange approach assuming that yaw angle does not affect the translational dynamics of the system which is clearly an approximation of the real behavior. Furthermore, the control law proposed in [9] is limited to the rotational dynamics and to the helicopter's altitude contrary to the present paper where the attitude and the position in three dimensions is addressed. In addition, the control of the helicopter's thrust is not bounded since it is obtained after a feedback linearization with the assumption of the boundedness of the pitch and roll angles.

In this paper, the asymptotic stabilization of both rotational and translational dynamics is considered. The proposed bounded control law is based on the global stabilization of multiple integrators with bounded control of [17], [13]. It is a simple control law with a very low computational cost which is crucial in real-time applications with limited embedded capabilities. The boundedness of all the control inputs of the quad-rotor helicopter is considered in addition to the positiveness of the thrust. The proposed control law is generalization of a previous result [8] obtained for the global stabilization of the PVTOL (Planar Vertical Take-Off and Landing) aircraft with bounded control. The PVTOL aircraft is the natural restriction of a V/STOL (Vertical/Short Take-Off and Landing) aircraft to a jet-borne maneuver in a vertical-lateral plane [11].

This paper is organized as follows. In section II, a brief description of the quad-rotor helicopter is given. The control law design is presented in section III and its stability is proved in section IV. Numerical simulations are provided in section V.

II. THE QUAD-ROTOR HELICOPTER MODEL

The quad-rotor helicopter developed at gipsa-lab's has four fixed-pitch rotors mounted at the four ends of a simple cross

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frame. Given that the front and rear motors rotate counter-clockwise while the other two rotate clockwise, gyroscopic effects and aerodynamic torques tend to cancel in trimmed flight. The thrust force is the sum of the vertical thrusts of each rotor. Pitch movement θ is obtained by increasing (reducing) the speed of the rear motor while reducing (increasing) the speed of the front motor. The roll movement ϕ is obtained similarly using the lateral motors. The yaw movement ψ is obtained by increasing (decreasing) the speed of the front and rear motors while decreasing (increasing) the speed of the lateral motors. This should be done while keeping the total thrust constant.

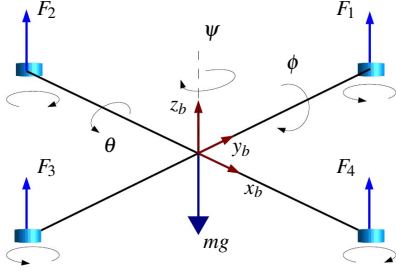


Fig. 2. 3D quad-rotor model

The equation of motion are written using the force and moment balance [1]

$$\begin{cases} \ddot{x} = u_1(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) - \frac{k_1}{m} \dot{x} \\ \ddot{y} = u_1(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) - \frac{k_2}{m} \dot{y} \\ \ddot{z} = -g + u_1(\cos \phi \cos \theta) - \frac{k_3}{m} \dot{z} \\ \ddot{\theta} = lu_2 - l\frac{k_4}{J_1} \dot{\theta} \\ \ddot{\phi} = lu_3 - l\frac{k_5}{J_2} \dot{\phi} \\ \ddot{\psi} = u_4 - \frac{k_6}{J_3} \dot{\psi} \end{cases} \quad (1)$$

with

$$\begin{aligned} u_1 &= (F_1 + F_2 + F_3 + F_4)/m \\ u_2 &= (F_1 - F_3)/J_1 \\ u_3 &= (-F_2 + F_4)/J_2 \\ u_4 &= C(F_1 - F_2 + F_3 - F_4)/J_3 \end{aligned} \quad (2)$$

x , y and z are the position coordinates of the centre of mass of the helicopter in a fixed inertial frame. The angles ϕ , θ and ψ represent respectively the roll, pitch and yaw angles. l is the distance from one rotor to the centre of mass of the helicopter. $J_{1,2,3}$ are the moment of inertia with respect to the axes and C is the force-to-moment scaling factor. $\{k_i\}_{i=1,\dots,6}$ are the drag coefficients. m is the mass of the quad-rotor helicopter and g is the gravitational acceleration. The $\{u_i\}_{i=1,\dots,4}$ are direct functions of the generated thrusts $\{F_i\}_{i=1,\dots,4}$ that can be controlled with local loops (low level inner control loops). Therefore, the $\{u_i\}_{i=1,\dots,4}$'s can directly be taken as control inputs. u_1 is the sum of the thrusts $F_{1,2,3,4}$ produced by each rotor. u_2 , u_3 and u_4 are respectively the pitch, the roll and the yaw inputs. Each generated thrust F_i is related to the rotational speed ω_i of the corresponding

rotor by the thrust coefficient k_r by the following equation:

$$F_i = k_r \omega_i^2 \quad i = 1, 2, 3, 4 \quad (3)$$

In practical application like the one treated in this paper, the rotor's rotational speed is bounded. As a result, the control inputs must be bounded as follows

$$\begin{aligned} 0 &\leq u_1 \leq \bar{u}_1 \\ -\bar{u}_2 &\leq u_2 \leq \bar{u}_2 \\ -\bar{u}_3 &\leq u_3 \leq \bar{u}_3 \\ -\bar{u}_4 &\leq u_4 \leq \bar{u}_4 \end{aligned} \quad (4)$$

where the \bar{u}_i 's can directly be deduced from the maximum voltage of the electrical drives used giving a maximum rotation speed and consequently a maximum thrust that one rotor can generate. In the present work, the normalized model of the quad-rotor is considered and the quad-rotor system (1) then writes:

$$\begin{cases} \ddot{x} = u(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) - k_x \dot{x} \\ \ddot{y} = u(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) - k_y \dot{y} \\ \ddot{z} = -1 + u(\cos \phi \cos \theta) - k_z \dot{z} \\ \ddot{\theta} = v_\theta - k_\theta \dot{\theta} \\ \ddot{\phi} = v_\phi - k_\phi \dot{\phi} \\ \ddot{\psi} = v_\psi - k_\psi \dot{\psi} \end{cases} \quad (5)$$

with

$$\begin{aligned} v_\theta &= u_2 l, & v_\phi &= u_3 l, & v_\psi &= u_4 \\ u &= \frac{u_1}{g}, & k_\theta &= \frac{l k_4}{J_1}, & k_\phi &= \frac{l k_5}{J_2}, \\ k_\psi &= \frac{k_6}{J_3}, & k_x &= \frac{k_1}{m}, & k_y &= \frac{k_2}{m}, \\ k_z &= \frac{k_3}{m} \end{aligned} \quad (6)$$

Our objective is to design a bounded stabilization control law for system (5). For this we neglect the drag coefficients $\{k_i\}_{i=1,\dots,6}$ (or equivalently we consider the system only at low speeds). The system becomes:

$$\begin{cases} \ddot{x} = u(\cos \phi \sin \theta \cos \psi + \sin \phi \sin \psi) \\ \ddot{y} = u(\cos \phi \sin \theta \sin \psi - \sin \phi \cos \psi) \\ \ddot{z} = -1 + u(\cos \phi \cos \theta) \\ \ddot{\theta} = v_\theta \\ \ddot{\phi} = v_\phi \\ \ddot{\psi} = v_\psi \end{cases} \quad (7)$$

System (7) will be used for the design of bounded control inputs verifying

$$\begin{aligned} 0 &\leq u \leq \bar{u} := \frac{\bar{u}_1}{g} \\ -l\bar{u}_2 &:= -\bar{v}_\theta \leq v_\theta \leq \bar{v}_\theta := l\bar{u}_2 \\ -l\bar{u}_3 &:= -\bar{v}_\phi \leq v_\phi \leq \bar{v}_\phi := l\bar{u}_3 \\ -\bar{u}_4 &:= -\bar{v}_\psi \leq v_\psi \leq \bar{v}_\psi := \bar{u}_4 \end{aligned} \quad (8)$$

but the obtained control law can be applied on the complete system with drag (5). The next section is devoted to the formulation of the control law.

III. FORMULATION OF THE CONTROL LAW

System (7) is clearly a cascade such that the evolution of the rotational dynamics is independent of the translational dynamics and since the translational dynamics depends on the angles. It is therefore natural to apply a cascaded design. The rotational dynamics will be first stabilized to a desired

configuration (determined in the following) and, in the same time, the angles θ , ϕ , and ψ will be taken as fictitious inputs for the translational dynamics in addition to the bounded positive input u . Therefore, system (7) is decomposed into two subsystems. The first subsystem Σ_t , composed of the three first equations, represents the translational dynamics in x , y and z . The second one Σ_r , composed of the last three equations, represents the rotational motion.

Assume now that one can stabilize the yaw dynamics (that is ψ and $\dot{\psi}$ with v_ψ as in (8), then after a sufficiently long time, system (7) will behave like the system:

$$\begin{cases} \ddot{x} = u \cos \phi \sin \theta \\ \ddot{y} = -u \sin \phi \\ \ddot{z} = -1 + u \cos \phi \cos \theta \\ \ddot{\theta} = v_\theta \\ \ddot{\phi} = v_\phi \end{cases} \quad (9)$$

Moreover, the right-hand side of (7) being bounded, the system can not diverge during that time. Therefore, finding bounded stabilizing control laws for (9) and for the yaw dynamics is equivalent to finding a bounded stabilizing control law for the quad-rotor helicopter (7).

A. Stabilization of the yaw dynamics

v_ψ can be chosen as in [12]:

$$v_\psi = \frac{\bar{v}_\psi}{\varepsilon_\psi + \varepsilon_\psi^2} (-\varepsilon_\psi \sigma(\psi) - \varepsilon_\psi^2 \sigma(\varepsilon_\psi \psi + \dot{\psi})) \quad (10)$$

where $0 < \varepsilon_\psi < 1$ is a free tuning parameter and $\sigma(\cdot)$ a saturation function as defined in Appendix I. This choice insures the asymptotic stabilization of $(\psi, \dot{\psi})$ [12].

B. Stabilization of system (9)

Let $p := (p_1, p_2, p_3, p_4, p_5, p_6) = (x, \dot{x}, y, \dot{y}, z, \dot{z})$ and $\eta := (\eta_1, \eta_2, \eta_3, \eta_4) = (\theta, \dot{\theta}, \phi, \dot{\phi})$ and rewrite the translational part Σ_t and rotational part Σ_r of (9) as follows:

$$\Sigma_t : \begin{cases} \dot{p}_1 = p_2 \\ \dot{p}_2 = u \cos \eta_3 \sin \eta_1 \\ \dot{p}_3 = p_4 \\ \dot{p}_4 = -u \sin \eta_3 \\ \dot{p}_5 = p_6 \\ \dot{p}_6 = -1 + u \cos \eta_3 \cos \eta_1 \end{cases} \quad \Sigma_r : \begin{cases} \dot{\eta}_1 = \eta_2 \\ \dot{\eta}_2 = v_\theta \\ \dot{\eta}_3 = \eta_4 \\ \dot{\eta}_4 = v_\phi \end{cases} \quad (11)$$

The idea is to choose bounded controls v_θ and v_ϕ that drive η_1 and η_3 to desired angles η_{1_d} and η_{3_d} . With an appropriate choice of these target configuration, it will be possible to transform Σ_t into three independent linear double integrators. For this, take:

$$\eta_{1_d} = \arctan \left(\frac{r_x}{1 + r_z} \right) \quad (12)$$

$$\eta_{3_d} = \arctan \left(\frac{-r_y}{\sqrt{r_x^2 + (1 + r_z)^2}} \right) \quad (13)$$

and choose as positive thrust u :

$$u = \sqrt{(1 + r_z)^2 + r_x^2 + r_y^2} \quad (14)$$

where r_x , r_y and r_z remain to be defined. Then Σ_t becomes:

$$\begin{cases} \ddot{p}_1 = r_x \\ \ddot{p}_3 = r_y \\ \ddot{p}_5 = r_z \end{cases} \quad (15)$$

Following [12], to insure that asymptotic stability of (15), we choose:

$$r_x = \frac{\bar{r}_x}{\varepsilon_x + \varepsilon_x^2} (-\varepsilon_x \sigma(p_2) - \varepsilon_x^2 \sigma(\varepsilon_x p_1 + p_2)) \quad (16)$$

$$r_y = \frac{\bar{r}_y}{\varepsilon_y + \varepsilon_y^2} (-\varepsilon_y \sigma(p_4) - \varepsilon_y^2 \sigma(\varepsilon_y p_3 + p_4)) \quad (17)$$

$$r_z = \frac{\bar{r}_z}{\varepsilon_z + \varepsilon_z^2} (-\varepsilon_z \sigma(p_6) - \varepsilon_z^2 \sigma(\varepsilon_z p_5 + p_6)) \quad (18)$$

where $0 < \varepsilon_{x,y,z} < 1$ are free tuning parameters, $\sigma(\cdot)$ is the saturation function defined in Appendix I and where the upper bounds \bar{r}_x , \bar{r}_y and \bar{r}_z are chosen such that:

$$\bar{u} = \sqrt{(1 + \bar{r}_z)^2 + \bar{r}_x^2 + \bar{r}_y^2} \quad (19)$$

Note that this implicitly implies that $\bar{u} > 1$ which is a natural constraint if one wants to compensate the effect of the gravity. And finally, to force η_1 and η_3 to converge to the desired angles η_{1_d} and η_{3_d} , one can take:

$$v_\theta = \frac{\bar{v}_\theta}{\beta_\theta + \varepsilon_\theta + \varepsilon_\theta^2} [\sigma_{\beta_\theta}(\dot{\eta}_{1_d}) - \varepsilon_\theta \sigma(\eta_2 - \dot{\eta}_{1_d}) - \varepsilon_\theta^2 \sigma(\varepsilon_\theta(\eta_1 - \eta_{1_d}) + (\eta_2 - \dot{\eta}_{1_d}))] \quad (20)$$

$$v_\phi = \frac{\bar{v}_\phi}{\beta_\phi + \varepsilon_\phi + \varepsilon_\phi^2} [\sigma_{\beta_\phi}(\dot{\eta}_{3_d}) - \varepsilon_\phi \sigma(\eta_4 - \dot{\eta}_{3_d}) - \varepsilon_\phi^2 \sigma(\varepsilon_\phi(\eta_3 - \eta_{3_d}) + (\eta_4 - \dot{\eta}_{3_d}))] \quad (21)$$

where, for any $\beta > 0$, $\sigma_\beta(\cdot) := \beta \sigma(\cdot)$ and where, as previously, $0 < \varepsilon_{\theta,\phi} < 1$ are free tuning parameters.

C. Main result

The following theorem summarizes the above construction:

Theorem 1: Consider the quad-rotor helicopter system (7) with input saturation bounds $\bar{u} > 1$ and $\bar{v}_\psi, \bar{v}_\phi, \bar{v}_\theta > 0$. Then the thrust input u given by (14) with v_ψ as in (10), v_θ as in (20) and v_ϕ as in (21) globally asymptotically stabilizes the quad-rotor helicopter normalized system to the origin.

The next section is devoted its formal proof.

IV. PROOF OF STABILITY RESULTS

The stability results follow from the cascaded design. The following proof breaks up into three main steps. First, the asymptotic stability of the yaw dynamics is established. Then, it is proved that η_1 and η_3 converge to the desired angles η_{1_d} and η_{3_d} . Finally, the global asymptotic stability of Σ_t is proved. The global asymptotic stability (GAS) of the overall system is then obtained by invoking [16].

A. GAS of the yaw dynamics

This is a direct application of Theorem 1 in [12].

B. GAS of $\eta_1 - \eta_{1_d}$ and $\eta_3 - \eta_{3_d}$

For $\psi = 0$, subsystem Σ_r is decomposed of two independent subsystems: Σ_{r_θ} related to θ and Σ_{r_ϕ} related to ϕ . The global asymptotic stability of Σ_{r_θ} is firstly considered.

$$\Sigma_{r_\theta} : \begin{cases} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= v_\theta \end{cases} \quad (22)$$

Let us first scale the system by taking $A = \frac{\beta_\theta + \varepsilon_\theta + \varepsilon_\theta^2}{\bar{v}_\theta}$: and apply the change of coordinates (parametrized by ε_θ determined later on):

$$\begin{cases} y_1 &= \varepsilon_\theta A \eta_1 + A \eta_2 \\ y_2 &= A \eta_2 \end{cases} \quad (23)$$

to obtain

$$\begin{cases} \dot{y}_1 &= \varepsilon_\theta y_2 + u_\theta \\ \dot{y}_2 &= u_\theta \end{cases} \quad (24)$$

where $u_\theta = A v_\theta$.

Finally, let $y_{1_d} := \varepsilon_\theta A \eta_{1_d} + A \dot{\eta}_{1_d}$, $y_{2_d} := A \dot{\eta}_{1_d}$ and $u_{\theta_d} := A \ddot{\eta}_{1_d}$. The control input u_θ then rewrites:

$$u_\theta = \sigma_{\beta_\theta}(u_{\theta_d}) - \varepsilon_\theta \sigma(y_2 - y_{2_d}) - \varepsilon_\theta^2 \sigma(y_1 - y_{1_d}) \quad (25)$$

Let us consider the Lyapunov function $V_2 = \frac{1}{2}(y_2 - y_{2_d})^2$. Then, $\dot{V}_2 = (y_2 - y_{2_d})(\dot{y}_2 - \dot{y}_{2_d}) = (y_2 - y_{2_d})(u_\theta - u_{\theta_d})$. Knowing that for all sufficiently small positive δ_θ , there exists a finite time instant t_{δ_θ} such that for all $t > t_{\delta_\theta}$, u_{θ_d} is bounded as follows (see Appendix II for details):

$$\left| u_{\theta_d} = \frac{d^2}{dt^2} \arctan\left(\frac{r_x}{1+r_z}\right) \right| < \delta_\theta \quad (26)$$

Therefore, if $|y_2 - y_{2_d}| > 1$ and $\beta_\theta + \delta_\theta + \varepsilon_\theta^2 < \varepsilon_\theta$, it follows that

$$|\sigma_{\beta_\theta}(u_{\theta_d}) - u_{\theta_d} - \varepsilon_\theta^2 \sigma(y_1 - y_{1_d})| < |\varepsilon_\theta \sigma(y_2 - y_{2_d})| \quad (27)$$

and therefore $u_\theta - u_{\theta_d}$ will be of opposite sign of $\sigma(y_2 - y_{2_d})$ and hence of $y_2 - y_{2_d}$. This ensures that $\dot{V}_2 < 0$ and as a result, $y_2 - y_{2_d}$ will join the interval $[-1, 1]$ after a finite time $t_1 > t_{\delta_\theta}$ and remains therein for all future t . During that time, invoking Lemma 4 of [12], $y_1 - y_{1_d}$ can not blow up. In $[-1, 1]$, u_θ takes the following form

$$u_\theta = \sigma_{\beta_\theta}(u_{\theta_d}) - \varepsilon_\theta (y_2 - y_{2_d}) - \varepsilon_\theta^2 \sigma(y_1 - y_{1_d}) \quad (28)$$

From equation (24), \dot{y}_1 can be written as

$$\dot{y}_1 = \varepsilon_\theta y_{2_d} + u_{\theta_d} + \sigma_{\beta_\theta}(u_{\theta_d}) - \varepsilon_\theta^2 \sigma(y_1 - y_{1_d}) - u_{\theta_d} \quad (29)$$

Let us consider the Lyapunov function $V_1 = \frac{1}{2}(y_1 - y_{1_d})^2$. In the case where $y_1 - y_{1_d}$ is not in $[-1, 1]$, imposing $\beta_\theta + \delta_\theta < \varepsilon_\theta^2$ will insure the decrease of V_1 . Then as for (27) one has

$$|\sigma_{\beta_\theta}(u_{\theta_d}) - u_{\theta_d}| < |\varepsilon_\theta^2 \sigma(y_1 - y_{1_d})| \quad (30)$$

which means that after a finite time $t_2 > t_1 > t_{\delta_\theta}$, $y_1 - y_{1_d}$ will join the interval $[-1, 1]$ where u_θ writes:

$$u_\theta = \sigma_{\beta_\theta}(u_{\theta_d}) - \varepsilon_\theta (y_2 - y_{2_d}) - \varepsilon_\theta^2 (y_1 - y_{1_d}) \quad (31)$$

$\beta_\theta > \delta_\theta$ is chosen, the control input u_θ takes the following form

$$u_\theta = u_{\theta_d} - \varepsilon_\theta (y_2 - y_{2_d}) - \varepsilon_\theta^2 (y_1 - y_{1_d}) \quad (32)$$

$\forall t > t_2$, the error system is hence given by:

$$\begin{cases} \dot{y}_1 - \dot{y}_{1_d} &= -\varepsilon_\theta^2 (y_1 - y_{1_d}) \\ \dot{y}_2 - \dot{y}_{2_d} &= -\varepsilon_\theta (y_2 - y_{2_d}) - \varepsilon_\theta^2 (y_1 - y_{1_d}) \end{cases}$$

which clearly gives that $\eta_1(t) \rightarrow \eta_{1_d}(t)$ as $t \rightarrow \infty$. A similar proof can be applied on Σ_{r_ϕ} and to obtain $\eta_3(t) \rightarrow \eta_{3_d}(t)$ as $t \rightarrow \infty$. To sum up, the choice of β_θ , β_ϕ , δ_θ , δ_ϕ , ε_θ and ε_ϕ must verify the following conditions.

$$\begin{aligned} \beta_\theta + \delta_\theta + \varepsilon_\theta^2 &< \varepsilon_\theta \\ \beta_\theta + \delta_\theta &< \varepsilon_\theta^2 \\ \delta_\theta &< \beta_\theta \\ \beta_\phi + \delta_\phi + \varepsilon_\phi^2 &< \varepsilon_\phi \\ \beta_\phi + \delta_\phi &< \varepsilon_\phi^2 \\ \delta_\phi &< \beta_\phi \end{aligned} \quad (33)$$

C. GAS of the translational dynamics

Subsystem Σ_t , when $\eta_1(t) = \eta_{1_d}(t)$ and $\eta_3(t) = \eta_{3_d}(t)$, takes the form (15) and is therefore globally asymptotically stabilized by (16-18) using Theorem 1 in [12].

D. GAS of the overall system

Knowing that the solutions of Σ_r are bounded, the Converging Input Bounded State (CIBS) property is fulfilled and the global asymptotic stability of Σ_t and Σ_r together is guaranteed for systems in cascade by [16]. This ends the proof

V. NUMERICAL SIMULATIONS

In this section, some simulation results are given. The nominal parameters of the quad-rotor are as in [20]

$$\begin{aligned} J_1 = J_2 &= 1.25 \text{Ns}^2/\text{rad} & J_3 &= 2.5 \text{Ns}^2/\text{rad} \\ kr &= 2.923 \times 10^{-3} & m &= 2 \text{kg} \\ k_1 = k_2 = k_3 &= 0.01 \text{Ns}/\text{m} & & \\ k_4 = k_5 = k_6 &= 0.012 \text{Ns}/\text{rad} & l &= 0.2 \\ g &= 9.81 \text{m}/\text{s}^2 & \bar{\omega} &= 604 \text{rad}/\text{s}^2 \end{aligned} \quad (34)$$

These numerical values are used in the normalized model (5). The initial state of the translational subsystem are $[x(0), y(0), z(0)] = [2, 2, 2]$ with zero linear velocity and $[\theta(0), \phi(0), \psi(0)] = [\frac{\pi}{2}, \frac{\pi}{2}, \frac{\pi}{4}]$ with zero angular velocities. The maximum bounds of \bar{r}_x , \bar{r}_y , \bar{r}_z , \bar{v}_θ , \bar{v}_ϕ , and \bar{v}_ψ can be calculated from the relation between the normalized bounds and the real bounds and the maximum rotor's speed $\bar{\omega}$ (equations (2), (3) and (6)). The numerical values used for the tuning parameters in the expressions of the control inputs are as follows: $\varepsilon_\psi = 1$, $\beta_\theta = \beta_\phi = 0.04$, $\varepsilon_\theta = \varepsilon_\phi = 0.9$, $\varepsilon_x = 0.3$, $\varepsilon_y = 0.3$ and $\varepsilon_z = 0.9$. The evolution of the states of translational subsystem is plotted on Fig.3 and Fig.4. The variations of the states of the rotational subsystem are plotted on Fig.5 and Fig.6. The bounded control inputs proposed in this paper are given in Fig.7. Finally, Fig.8 shows the path followed by the helicopter in three dimensions.

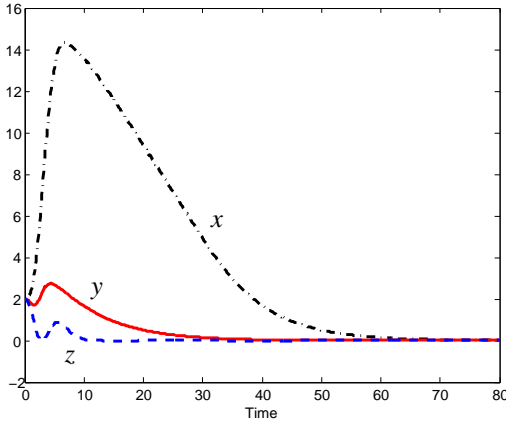


Fig. 3. Position of the normalized quad-rotor helicopter with drag

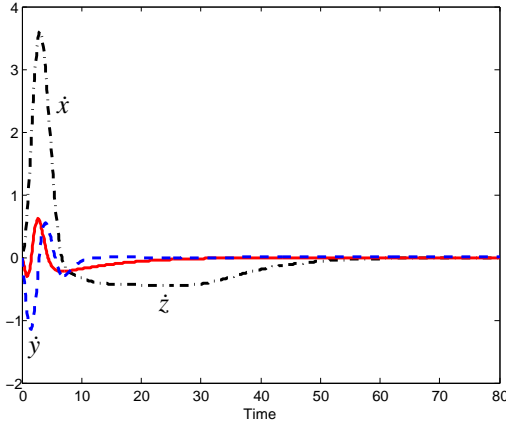


Fig. 4. Linear velocities of the normalized quad-rotor helicopter with drag

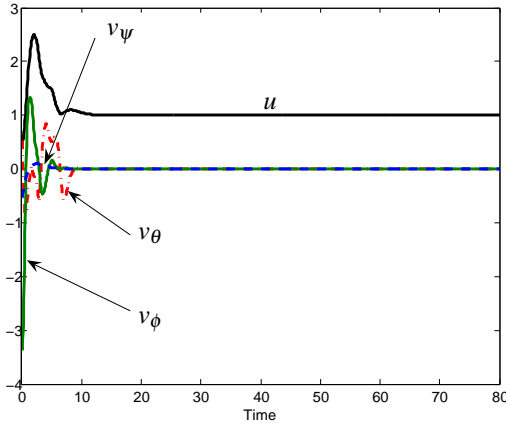


Fig. 7. Bounded inputs applied to the normalized quad-rotor helicopter with drag

VI. CONCLUSIONS AND FUTURE WORK

In this paper, a global stabilization of a quad-rotor helicopter is presented. It is an under-actuated system with six degrees of freedom and only four inputs. The system is divided into two subsystems: rotational and translational. The control scheme is based on the cascaded design for nonlinear systems. To respect the boundedness of the control

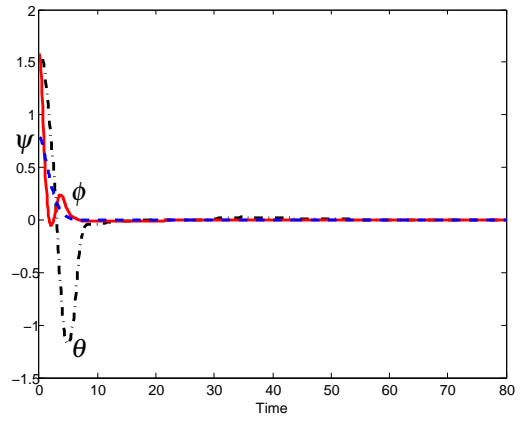


Fig. 5. Orientation of the normalized quad-rotor helicopter with drag

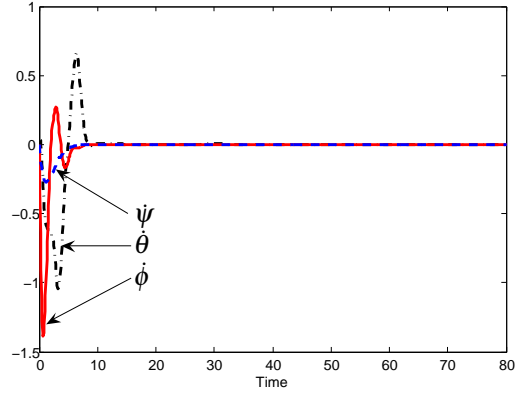


Fig. 6. Angular velocities of the normalized quad-rotor helicopter with drag

inputs while respecting the simplicity of the controls, the proposed control design exploits the technique based on the sum of saturating functions. For the future work, the proposed approach will be implemented on the platform of Fig.1 to check the practical applicability of the control law. This approach will be compared with other control schemes such as the nonlinear model predictive control or backstepping. As far as the authors know, the proposed approach is by far the simplest and the more suitable for embedded implementation.

APPENDIX I THE SATURATION FUNCTION

Let $\sigma_M(\cdot)$ be the following twice-differentiable function bounded between $+M$ and $-M$ parameterized by $0 < \alpha < 1$ defined as follows

$$\sigma_M(s) = \begin{cases} -M & \text{si } s < -1 - \alpha \\ p_1(s) = a_1 s^2 + b_1 s + c_1 & \text{si } s \in [-M - \alpha, -M + \alpha] \\ s & \text{si } s \in [-M + \alpha, M + \alpha] \\ p_2(s) = a_2 s^2 + b_2 s - c_2 & \text{si } s \in]M - \alpha, M + \alpha] \\ +M & \text{si } s > 1 + \alpha \end{cases} \quad (35)$$

with $a_1 = -a_2 = \frac{1}{4\alpha}$, $b_1 = b_2 = \frac{1}{2} + \frac{M}{2\alpha}$, $c_1 = -M + \frac{\alpha}{2} + b_1 M - a_1(M^2 + \alpha^2)$ and $c_2 = M - \frac{\alpha}{2} - b_2 M - a_2(M^2 + \alpha^2)$

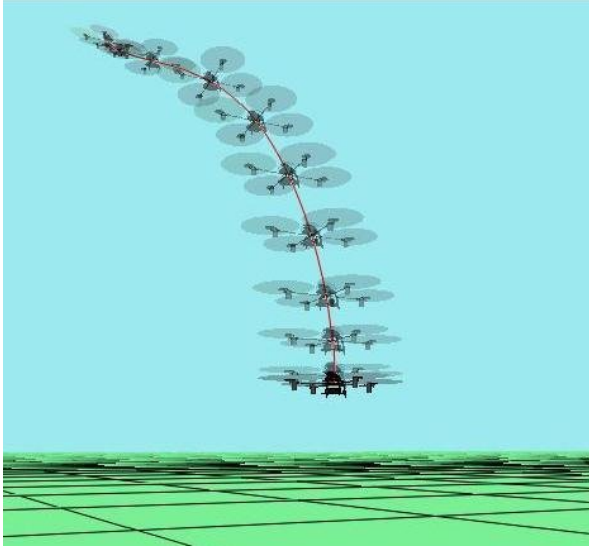


Fig. 8. 3D trajectory of the four-rotor helicopter

to insure the twice differentiability of σ_M . In this paper, we apply $\sigma_M(\cdot)$ (M is omitted if $M = 1$) with the saturation level $M = 1$ (M is omitted if $M = 1$). Note that when $\alpha \rightarrow 0$, $\sigma(\cdot)$ tends to the classical saturation function $\text{sat}(\cdot) = \text{sign}(\cdot)|\cdot|$.

APPENDIX II

BOUNDEDNESS OF η_{1_d} AND η_{3_d} AND THEIR RESPECTIVE TIME DERIVATIVES

The first and second time derivatives of η_{1_d} and η_{3_d} are:

$$\begin{aligned}\dot{\eta}_{1_d} &= \frac{\dot{r}_x(1+r_z) - r_x \dot{r}_z}{(1+r_z)^2 + r_x^2} \\ \dot{\eta}_{1_d} &= \frac{\ddot{r}_x(1+r_z) - r_x \ddot{r}_z}{(1+r_z)^2 + r_x^2} \\ &\quad - \frac{2(\dot{r}_x(1+r_z) - r_x \dot{r}_z)((1+r_z)\dot{r}_z + r_x \dot{r}_x)}{((1+r_z)^2 + r_x^2)^2} \\ &\quad - \frac{\dot{r}_y \sqrt{(1+r_z)^2 + r_x^2} + r_y \left(\frac{\dot{r}_x r_x + (1+r_z)\dot{r}_z}{\sqrt{(1+r_z)^2 + r_x^2}} \right)}{((1+r_z)^2 + r_x^2)^{3/2}} \\ \dot{\eta}_{3_d} &= \frac{-\ddot{r}_y \sqrt{r_x^2 + (1+r_z)^2} u^2}{u^2} \\ \dot{\eta}_{3_d} &= \frac{r_y [\dot{r}_x r_x + (\dot{r}_x)^2 + (\dot{r}_z)^2 + (1+r_z)\ddot{r}_z] [r_x^2 + (1+r_z)^2]}{u^2 (r_x^2 + (1+r_z)^2)^{3/2}} \\ &\quad - \frac{r_y (\dot{r}_x r_x + (1+r_z)\dot{r}_z)^2}{u^2 (r_x^2 + (1+r_z)^2)^{3/2}}\end{aligned}$$

where, r_x , r_y , and r_z are:

$$\begin{aligned}\dot{r}_x &= \frac{\bar{r}_x}{\varepsilon_x + \varepsilon_x^2} (-\varepsilon_x w_1 \dot{\sigma}(p_2) - \varepsilon_x^2 (\varepsilon_x p_2 + w_1) \dot{\sigma}(\varepsilon_x p_1 + p_2)) \\ \dot{r}_y &= \frac{\bar{r}_y}{\varepsilon_y + \varepsilon_y^2} (-\varepsilon_y w_2 \dot{\sigma}(p_4) - \varepsilon_y^2 (\varepsilon_y p_4 + w_2) \dot{\sigma}(\varepsilon_y p_3 + p_4)) \\ \dot{r}_z &= \frac{\bar{r}_z}{\varepsilon_z + \varepsilon_z^2} (-\varepsilon_z w_3 \dot{\sigma}(p_6) - \varepsilon_z^2 (\varepsilon_z p_6 + w_3) \dot{\sigma}(\varepsilon_z p_5 + p_6))\end{aligned}$$

and

$$\begin{aligned}w_1 &= u \cos \eta_3 \sin \eta_1 \\ w_2 &= -u \sin \eta_3 \\ w_3 &= -1 + u \cos \eta_3 \cos \eta_1\end{aligned}$$

The first derivatives of r_x , r_y and r_z are composed of bounded terms: $\sigma(\cdot)$ is twice differentiable and this ensures the boundedness of $\dot{\sigma}$ and $\ddot{\sigma}$ and w_1 , w_2 and w_3 are products of bounded trigonometric functions and u (also bounded). Consequently, $\dot{\eta}_{1_d}$ and $\dot{\eta}_{3_d}$ are bounded. For $\ddot{\eta}_{1_d}$ and $\ddot{\eta}_{3_d}$,

the same reasoning can be applied with the application of the result proved in [21] for the global stabilization of the PVTOL aircraft rotational dynamics. Note that reducing \bar{r}_x , \bar{r}_y and \bar{r}_z also reduces the bounds on $\ddot{\eta}_{1_d}$ and $\ddot{\eta}_{3_d}$, that is on u_{θ_d} and u_{ϕ_d} .

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